Lesson 2.3 Exercises, pages 114–121

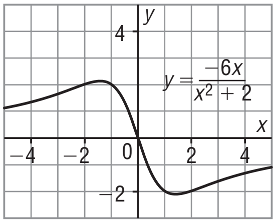
A

4. For the graph of each rational function below:

i) Write the equations of any asymptotes.

ii) State the domain.

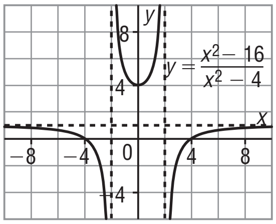
a)



i) There is no vertical asymptote.
The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.

ii) Since all real values of x are permissible, the domain is: $x \in \mathbb{R}$

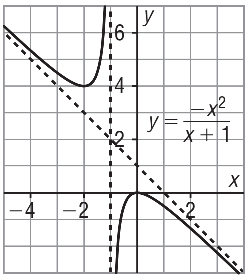
b)



i) The vertical asymptotes have equations $x = -2$ and $x = 2$. The horizontal asymptote has equation $y = 1$.

ii) The domain is: $x \neq \pm 2$

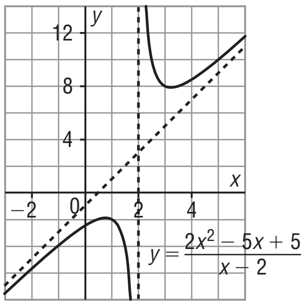
c)



i) The vertical asymptote has equation $x = -1$.
The oblique asymptote has slope -1 and y-intercept 1, so its equation is $y = -x + 1$.

ii) The domain is: $x \neq -1$

d)



i) The vertical asymptote has equation $x = 2$.
The oblique asymptote has slope 2 and y-intercept -1, so its equation is $y = 2x - 1$.

ii) The domain is: $x \neq 2$

B

5. For the graph of each rational function:

i) Write the coordinates of any hole.

ii) Write the equations of any vertical asymptotes.

a) $y = \frac{x^2 - 4}{x + 2}$

The function is undefined when $x + 2 = 0$; that is, when $x = -2$.

i) Factor: $y = \frac{(x + 2)(x - 2)}{x + 2}$

There is a hole at $x = -2$ on the line with equation $y = x - 2$.

The coordinates of the hole are: $(-2, -4)$

ii) There is no vertical asymptote.

b) $y = \frac{x^2 + x - 2}{x^2 - 2x - 3}$

The function is undefined when:

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

i) Factor: $y = \frac{(x + 2)(x - 1)}{(x - 3)(x + 1)}$

There is no hole.

ii) The vertical asymptotes have equations: $x = 3$ and $x = -1$

c) $y = \frac{x^2 - 4}{x^2 + 4}$

The function is undefined when $x^2 + 4 = 0$.

Since $x^2 + 4$ is never equal to 0, the function is defined for all real values of x.

i) There is no hole.

ii) There is no vertical asymptote.

d) $y = \frac{x^2 - 5x + 4}{x - 1}$

The function is undefined when $x - 1 = 0$, or $x = 1$.

i) Factor: $y = \frac{(x - 4)(x - 1)}{x - 1}$

There is a hole at $x = 1$ on the line with equation $y = x - 4$.

The coordinates of the hole are: $(1, -3)$

ii) There is no vertical asymptote.

6. For each rational function, determine whether its graph has a horizontal asymptote. If it does, write its equation.

a) $y = \frac{4x}{x + 2}$

The degrees of the numerator and denominator are equal, so there is a horizontal asymptote.

The leading coefficients of the numerator and denominator are 4 and 1 respectively. So, the horizontal asymptote has equation: $y = 4$

b) $y = \frac{x^2 - 16}{x^2 + 4}$

The degrees of the numerator and denominator are equal, so there is a horizontal asymptote.

Both leading coefficients are 1.

So, the horizontal asymptote has equation: $y = 1$

c) $y = \dfrac{x}{x^2 - 25}$

The degree of the numerator is less than the degree of the denominator, so there is a horizontal asymptote with equation $y = 0$.

d) $y = \dfrac{x - 2}{x - 4}$

The degrees of the numerator and denominator are equal, so there is a horizontal asymptote. Both leading coefficients are 1. So, the horizontal asymptote has equation: $y = 1$

7. For each rational function, determine whether its graph has an oblique asymptote. If it does, write its equation.

a) $y = \dfrac{x^2 - 2x - 5}{x - 1}$

The numerator does not factor. The degree of the numerator is 1 more than the degree of the denominator, so there is an oblique asymptote. Determine:
$(x^2 - 2x - 5) \div (x - 1)$

$$\begin{array}{r|rrr} 1 & 1 & -2 & -5 \\ & & 1 & -1 \\ \hline & 1 & -1 & -6 \end{array}$$

The quotient is $x - 1$; so the equation of the oblique asymptote is: $y = x - 1$

b) $y = \dfrac{x^2 - 7x + 10}{x - 5}$

Factor the numerator:
$y = \dfrac{(x - 5)(x - 2)}{x - 5}$
After removing a common factor, the equation is:
$y = x - 2, x \neq 5$
So, there is no oblique asymptote.

c) $y = \dfrac{x^2}{4 - x}$

There are no common factors. The degree of the numerator is 1 more than the degree of the denominator, so there is an oblique asymptote.
Determine: $x^2 \div (4 - x)$
Write: $-x^2 \div (x - 4)$

$$\begin{array}{r|rrr} 4 & -1 & 0 & 0 \\ & & -4 & -16 \\ \hline & -1 & -4 & -16 \end{array}$$

The quotient is $-x - 4$; so the equation of the oblique asymptote is: $y = -x - 4$

d) $y = \dfrac{-2x^2 + 3x + 1}{x + 2}$

There are no common factors. The degree of the numerator is 1 more than the degree of the denominator, so there is an oblique asymptote.
Determine:
$(-2x^2 + 3x + 1) \div (x + 2)$

$$\begin{array}{r|rrr} -2 & -2 & 3 & 1 \\ & & 4 & -14 \\ \hline & -2 & 7 & -13 \end{array}$$

The quotient is $-2x + 7$; so the equation of the oblique asymptote is: $y = -2x + 7$

8. Solve each rational equation by graphing.
Give the solutions to the nearest tenth where necessary.

a) $\dfrac{4}{x + 2} + 1 = 0$

Graph a related function:
$f(x) = \dfrac{4}{x + 2} + 1$
Use graphing technology to determine the zero:
$x = -6$

b) $\dfrac{x - 2}{x - 4} = x + 3$

Graph a related function:
$f(x) = \dfrac{x - 2}{x - 4} - x - 3$
Use graphing technology to determine the zeros:
$x \doteq -2.3$ or $x \doteq 4.3$

c) $\dfrac{x^2 - x - 2}{x^2 - 4} = x + 6$

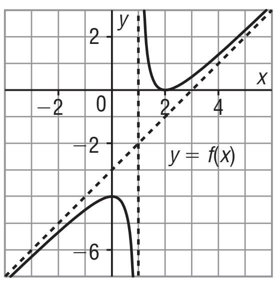
Graph a related function:
$f(x) = \dfrac{x^2 - x - 2}{x^2 - 4} - x - 6$
Use graphing technology to determine the zeros:
$x \doteq -4.6$ or $x \doteq -2.4$

d) $\dfrac{4}{3x^2 - 1} = 2 + \dfrac{10}{6x - 1}$

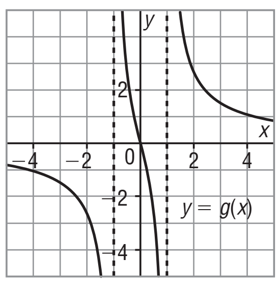
Graph a related function:
$f(x) = \dfrac{4}{3x^2 - 1} - 2 - \dfrac{10}{6x - 1}$
Use graphing technology to determine the zeros:
$x \doteq -1.3$ or $x \doteq -0.1$ or $x \doteq 0.8$

9. Match each function to its graph. Justify your choice.

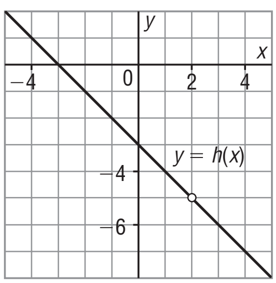
i) Graph A



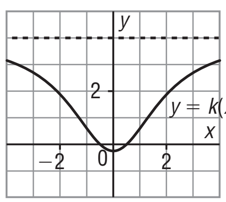
ii) Graph B



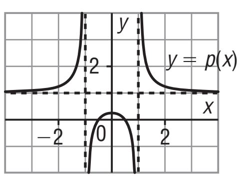
iii) Graph C



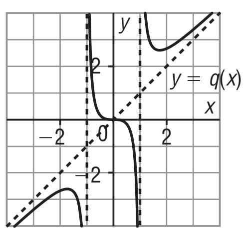
iv) Graph D



v) Graph E



vi) Graph F



a) $y = \frac{x^2 + x - 6}{2 - x}$

Factor:

$y = \frac{(x + 3)(x - 2)}{2 - x}$

Rewrite as:

$y = \frac{-(x + 3)(x - 2)}{x - 2}$

The graph is a line with a hole at $x = 2$.
The function matches Graph C.

b) $y = \frac{x^2 - 4x + 4}{x - 1}$

Factor:

$y = \frac{(x - 2)(x - 2)}{x - 1}$

There are no common factors.
There is a vertical asymptote at $x = 1$.
The degree of the numerator is 1 more than the degree of the denominator, so there is an oblique asymptote.
The function matches Graph A.

c) $y = \frac{x^3}{x^2 - 1}$

Factor:

$y = \frac{x^3}{(x - 1)(x + 1)}$

There are vertical asymptotes at $x = 1$ and $x = -1$.
The degree of the numerator is 1 more than the degree of the denominator, so there is an oblique asymptote.
The function matches Graph F.

d) $y = \frac{4x}{x^2 - 1}$

Factor:

$y = \frac{4x}{(x - 1)(x + 1)}$

There are no common factors.
There are vertical asymptotes at $x = 1$ and $x = -1$.
The degree of the numerator is less than the degree of the denominator, so there is a horizontal asymptote at $y = 0$.
The function matches Graph B.

e) $y = \frac{4x^2 - 1}{x^2 + 4}$

The denominator is always positive, so the graph has no hole or vertical asymptote.
The degrees of the numerator and denominator are equal, so the graph has a horizontal asymptote that is not the x-axis.
The function matches Graph D.

f) $y = \frac{4x^2 - 1}{4x^2 - 4}$

Factor:

$y = \frac{(2x + 1)(2x - 1)}{4(x - 1)(x + 1)}$

There are no common factors.
There are vertical asymptotes at $x = 1$ and $x = -1$.
The degrees of the numerator and denominator are equal, so the graph has a horizontal asymptote that is not the x-axis.
The function matches Graph E.

10. For the graph of each function:

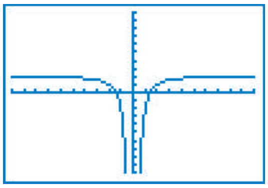
i) Determine the equations of any asymptotes and the coordinates of any hole.

ii) Determine the domain.

iii) Use graphing technology to verify the characteristics, and to explain the behaviour of the graph close to the non-permissible values.

a) $y = \dfrac{2x^2 - 4}{x^2}$

i) The function is undefined when $x^2 = 0$; that is, when $x = 0$. There are no common factors, so there is a vertical asymptote with equation $x = 0$. The degrees of the numerator and denominator are equal, so there is a horizontal asymptote. The leading coefficients of the numerator and denominator are 2 and 1, respectively. So, the horizontal asymptote has equation: $y = 2$



ii) The domain is: $x \neq 0$

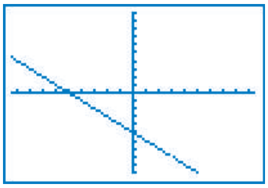
iii) From the calculator screen: as $|x| \to \infty$, $y \to 2$, which verifies the horizontal asymptote; as $x \to 0$, $y \to \infty$, which verifies the vertical asymptote

b) $y = \dfrac{x^2 + 2x - 15}{3 - x}$

i) The function is undefined when $3 - x = 0$; that is, when $x = 3$.

Factor: $y = \dfrac{(x + 5)(x - 3)}{3 - x}$, or $y = \dfrac{-(x + 5)(x - 3)}{x - 3}$

$(x - 3)$ is a common factor, so there is a hole at $x = 3$.



The function is: $y = -x - 5, x \neq 3$

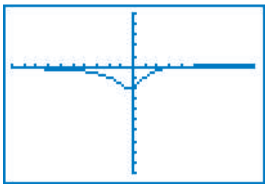
The coordinates of the hole are: $(3, -8)$

ii) The domain is: $x \neq 3$

iii) From the calculator screen: as $x \to 3$, $y \to -8$, which verifies the hole

c) $y = \dfrac{x - 4}{x^2 + 2}$

i) The denominator is always positive, so there are no restrictions on x, and there is no hole or vertical asymptote. Since the degree of the numerator is less than the degree of the denominator, there is a horizontal asymptote at $y = 0$.



ii) The domain is: $x \in \mathbb{R}$

iii) From the calculator screen: as $|x| \to \infty$, $y \to 0$, which verifies the horizontal asymptote

11. The speed of a boat in still water is 10 km/h. It travels 25 km upstream and 25 km downstream in 6 h. This equation models the total time for the journey in terms of the speed of the current, v kilometres per hour: $\frac{25}{10 - v} + \frac{25}{10 + v} = 6$
What is the speed of the current, to the nearest whole number?

Write a related function: $f(v) = \frac{25}{10 - v} + \frac{25}{10 + v} - 6$
Use graphing technology to determine the zeros:
$x \doteq -4.1$ or $x \doteq 4.1$
Ignore the negative root because speed cannot be negative.
To the nearest whole number, the speed of the current is 4 km/h.

C

12. Create an equation for a rational function whose graph has the given characteristics.

a) The graph is the line $y = x$ with two holes.

For a graph to have 2 holes, the numerator and denominator must have 2 different binomial common factors. Begin with the equation of the line, $y = x$. Choose two binomial factors, such as $(x + 3)$ and $(x - 4)$. Multiply and divide x by these factors. A possible function is
$y = \frac{x(x + 3)(x - 4)}{(x + 3)(x - 4)}$.

b) The graph has a horizontal asymptote with equation $y = -6$, and no vertical asymptotes.

For a horizontal asymptote $y = -6$, the function must approach -6 as $|x|$ approaches infinity and the degrees of the numerator and denominator must be equal. The leading coefficient of the numerator could be -6, then the leading coefficient of the denominator would be 1. For no vertical asymptotes, the denominator must never be 0.

A possible function is $y = \frac{-6x^2}{x^2 + 1}$.

c) The graph has an oblique asymptote with equation $y = x + 1$, and a vertical asymptote with equation $x = 2$.

For an oblique asymptote, when the denominator is divided into the numerator, the quotient must be $x + 1$ and there must be a remainder. For a vertical asymptote with equation $x = 2$, the denominator contains the factor $(x - 2)$ and the numerator does not.

A possible function is $y = \frac{(x + 1)(x - 2) + 3}{x - 2}$.

d) The graph has two vertical asymptotes, a horizontal asymptote that is not the x-axis, two holes, and:

i) the y-axis is a line of symmetry

ii) the y-axis is not a line of symmetry

i) For the y-axis to be a line of symmetry, the function must have the same value for $x = a$ and $x = -a$, so the numerator and denominator are the products of factors of the form $(x - a)(x + a)$, or $x^2 - a^2$. For two vertical asymptotes, the denominator contains two binomial factors, with opposite constant terms, such as $(x - 3)(x + 3)$, or $x^2 - 9$. For a horizontal asymptote such as $y = 2$, the leading coefficient of the numerator could be 2, then the leading coefficient of the denominator would be 1, and the term in the numerator would be $2x^2$. For two holes, the numerator and denominator must have 2 different binomial common factors, with opposite constant terms, such as $(x - 1)(x + 1)$, or $x^2 - 1$.

A possible function is $y = \dfrac{2x^2(x^2 - 1)}{(x^2 - 1)(x^2 - 9)}$.

ii) For the graph to have no symmetry about the y-axis, replace the factor x^2 in the numerator with a factor not of the form $(x^2 - a^2)$, for example, $(x^2 + x)$. A possible function is

$$y = \frac{2(x^2 + x)(x^2 - 1)}{(x^2 - 1)(x^2 - 9)}.$$