Lesson 7.3 Exercises, pages 611–618

A

3. Write each expression in terms of a single trigonometric function.

a) $\frac{\cos\theta}{\sin\theta}$

$= \cot\theta$

b) $\frac{\sin^2\theta}{\cos^2\theta}$

$= \tan^2\theta$

c) $\sin^2\theta \sec\theta \cos\theta \csc\theta$

$= (\sin^2\theta)\left(\frac{1}{\cos\theta}\right)(\cos\theta)\left(\frac{1}{\sin\theta}\right)$

$= \sin\theta$

d) $\frac{\sin^2\theta}{\tan^2\theta}$

$= \frac{\sin^2\theta}{\frac{\sin^2\theta}{\cos^2\theta}}$

$= \cos^2\theta$

4. Determine the non-permissible values of θ.

a) $\sec\theta$

$\sec\theta = \frac{1}{\cos\theta}$, so non-permissible values occur when $\cos\theta = 0$, so $\theta = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

b) $\tan\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, so non-permissible values occur when $\cos\theta = 0$, so $\theta = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

c) $\frac{\csc\theta}{\cos\theta}$

$\csc\theta = \frac{1}{\sin\theta}$, so non-permissible values occur when $\sin\theta = 0$, so $\theta = \pi k$, $k \in \mathbb{Z}$; or when $\cos\theta = 0$, so $\theta = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

d) $\frac{\sec\theta}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$, so non-permissible values occur when $\cos\theta = 0$, so $\theta = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$; or when $\sin\theta = 0$, so $\theta = \pi k, k \in \mathbb{Z}$

5. Verify each identity for the given value of θ.

a) $\tan\theta \csc\theta \sec\theta = \sec^2\theta; \theta = 150°$

Substitute: $\theta = 150°$

$$\begin{aligned}
\text{L.S.} &= \tan\theta \csc\theta \sec\theta \\
&= (\tan 150°)(\csc 150°)(\sec 150°) \\
&= (-\tan 30°)\left(\frac{1}{\sin 150°}\right)\left(\frac{1}{\cos 150°}\right) \\
&= (-\tan 30°)\left(\frac{1}{\sin 30°}\right)\left(\frac{-1}{\cos 30°}\right) \\
&= \left(\frac{-1}{\sqrt{3}}\right)(2)\left(\frac{-2}{\sqrt{3}}\right) \\
&= \frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
\text{R.S.} &= \sec^2\theta \\
&= \sec^2(150°) \\
&= \frac{1}{\cos^2(150°)} \\
&= \left(\frac{-1}{\cos 30°}\right)^2 \\
&= \left(\frac{-2}{\sqrt{3}}\right)^2 \\
&= \frac{4}{3}
\end{aligned}$$

The left side is equal to the right side, so $\theta = 150°$ is verified.

b) $\dfrac{\tan \theta \csc^2\theta}{\sec^2\theta} = \cot \theta;\ \theta = \dfrac{4\pi}{3}$

Substitute: $\theta = \dfrac{4\pi}{3}$

$$\text{L.S.} = \frac{\tan \theta \csc^2\theta}{\sec^2\theta}$$
$$= \frac{\left(\tan \frac{4\pi}{3}\right)\left(\csc^2\left(\frac{4\pi}{3}\right)\right)}{\sec^2\left(\frac{4\pi}{3}\right)}$$
$$= \frac{\left(\tan \frac{\pi}{3}\right)\left(-\cos \frac{\pi}{3}\right)^2}{\left(-\sin \frac{\pi}{3}\right)^2}$$
$$= \frac{(\sqrt{3})\left(-\frac{1}{2}\right)^2}{\left(-\frac{\sqrt{3}}{2}\right)^2}$$
$$= \frac{1}{\sqrt{3}}$$

$$\text{R.S.} = \cot \theta$$
$$= \cot \frac{4\pi}{3}$$
$$= \frac{1}{\tan \frac{\pi}{3}}$$
$$= \frac{1}{\sqrt{3}}$$

The left side is equal to the right side, so $\theta = \dfrac{4\pi}{3}$ is verified.

B

6. Prove each identity in question 5.

a) $\tan \theta \csc \theta \sec \theta = \sec^2\theta$

$$\text{L.S.} = \tan \theta \csc \theta \sec \theta$$
$$= \left(\frac{\sin \theta}{\cos \theta}\right)\left(\frac{1}{\sin \theta}\right)\left(\frac{1}{\cos \theta}\right)$$
$$= \frac{1}{\cos^2\theta}$$
$$= \sec^2\theta$$
$$= \text{R.S.}$$

The left side is equal to the right side, so the identity is proved.

b) $\dfrac{\tan \theta \csc^2\theta}{\sec^2\theta} = \cot \theta$

$$\text{L.S.} = \frac{\tan \theta \csc^2\theta}{\sec^2\theta}$$
$$= \frac{\left(\frac{\sin \theta}{\cos \theta}\right)\left(\frac{1}{\sin \theta}\right)^2}{\left(\frac{1}{\cos \theta}\right)^2}$$
$$= \frac{\frac{1}{(\sin \theta)(\cos \theta)}}{\left(\frac{1}{\cos \theta}\right)^2}$$
$$= \frac{\cos \theta}{\sin \theta}$$
$$= \cot \theta$$
$$= \text{R.S.}$$

The left side is equal to the right side, so the identity is proved.

7. For each identity:

i) Verify the identity using graphing technology.

ii) Prove the identity.

a) $1 - \sin\theta = (\sin\theta)(\csc\theta - 1)$

i) Graph: $y = 1 - \sin\theta$ and $y = \sin\theta\left(\frac{1}{\sin\theta} - 1\right)$

The graphs coincide, so the identity is verified.

ii) $$\begin{aligned} \text{R.S.} &= (\sin\theta)(\csc\theta - 1) \\ &= (\sin\theta)\left(\frac{1}{\sin\theta} - 1\right) \\ &= 1 - \sin\theta \\ &= \text{L.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

b) $-\cot\theta = \dfrac{1 - \cot\theta}{1 - \tan\theta}$

i) Graph: $y = \dfrac{-1}{\tan\theta}$ and $y = \dfrac{1 - \frac{1}{\tan\theta}}{1 - \tan\theta}$

The graphs coincide, so the identity is verified.

ii) $$\begin{aligned} \text{R.S.} &= \frac{1 - \cot\theta}{1 - \tan\theta} \\ &= \frac{1 - \frac{1}{\tan\theta}}{1 - \tan\theta} \\ &= \frac{\tan\theta - 1}{(\tan\theta)(1 - \tan\theta)} \\ &= \frac{-1}{\tan\theta} \\ &= -\cot\theta \\ &= \text{L.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

8. For each identity:

i) Verify the identity for $\theta = 45°$.

ii) Prove the identity.

a) $\dfrac{\cot\theta}{\cos\theta} - \csc\theta = 0$

i) Substitute: $\theta = 45°$

$$\begin{aligned} \text{L.S.} &= \frac{\cot\theta}{\cos\theta} - \csc\theta \\ &= \frac{\cot 45°}{\cos 45°} - \csc 45° \\ &= \frac{1}{\frac{1}{\sqrt{2}}} - \sqrt{2} \\ &= \sqrt{2} - \sqrt{2} \\ &= 0 \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so $\theta = 45°$ is verified.

b) $\tan^2\theta\cos^2\theta + \sin^2\theta = \dfrac{2}{\csc^2\theta}$

i) Substitute: $\theta = 45°$

$$\begin{aligned} \text{L.S.} &= \tan^2\theta\cos^2\theta + \sin^2\theta \\ &= (\tan 45°)^2(\cos 45°)^2 + (\sin 45°)^2 \\ &= (1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{R.S.} &= \frac{2}{\csc^2\theta} \\ &= \frac{2}{(\csc 45°)^2} \\ &= \frac{2}{(\sqrt{2})^2} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

The left side is equal to the right side, so $\theta = 45°$ is verified.

ii) L.S. $= \dfrac{\cot\theta}{\cos\theta} - \csc\theta$

$= \dfrac{\frac{\cos\theta}{\sin\theta}}{\cos\theta} - \dfrac{1}{\sin\theta}$

$= \dfrac{1}{\sin\theta} - \dfrac{1}{\sin\theta}$

$= 0$

$=$ R.S.

The left side is equal to the right side, so the identity is proved.

ii) L.S. $= \tan^2\theta\cos^2\theta + \sin^2\theta$

$= \left(\dfrac{\sin^2\theta}{\cos^2\theta}\right)(\cos^2\theta) + \sin^2\theta$

$= \sin^2\theta + \sin^2\theta$

$= 2\sin^2\theta$

$= \dfrac{2}{\csc^2\theta}$

$=$ R.S.

The left side is equal to the right side, so the identity is proved.

9. For each identity:

i) Verify the identity for $\theta = \dfrac{7\pi}{6}$.

ii) Prove the identity.

a) $\csc\theta = \dfrac{\csc\theta - 1}{1 - \sin\theta}$

i) Substitute: $\theta = \dfrac{7\pi}{6}$

L.S. $= \csc\theta$

$= \csc\dfrac{7\pi}{6}$

$= -\csc\dfrac{\pi}{6}$

$= -2$

R.S. $= \dfrac{\csc\theta - 1}{1 - \sin\theta}$

$= \dfrac{\csc\frac{7\pi}{6} - 1}{1 - \sin\frac{7\pi}{6}}$

$= \dfrac{-2 - 1}{1 - \left(-\frac{1}{2}\right)}$

$= -2$

The left side is equal to the right side, so $\theta = \dfrac{7\pi}{6}$ is verified.

ii) R.S. $= \dfrac{\csc\theta - 1}{1 - \sin\theta}$

$= \dfrac{\frac{1}{\sin\theta} - 1}{1 - \sin\theta}$

$= \dfrac{1 - \sin\theta}{(\sin\theta)(1 - \sin\theta)}$

$= \dfrac{1}{\sin\theta}$

$= \csc\theta$

$=$ L.S.

The left side is equal to the right side, so the identity is proved.

b) $\dfrac{\cos\theta - \cot\theta}{1 - \sin\theta} = -\cot\theta$

i) Substitute: $\theta = \dfrac{7\pi}{6}$

L.S. $= \dfrac{\cos\theta - \cot\theta}{1 - \sin\theta}$

$= \dfrac{\cos\frac{7\pi}{6} - \cot\frac{7\pi}{6}}{1 - \sin\frac{7\pi}{6}}$

$= \dfrac{-\frac{\sqrt{3}}{2} - \sqrt{3}}{1 - \left(-\frac{1}{2}\right)}$

$= -\sqrt{3}$

R.S. $= -\cot\theta$

$= -\cot\dfrac{7\pi}{6}$

$= -\sqrt{3}$

The left side is equal to the right side, so $\theta = \dfrac{7\pi}{6}$ is verified.

ii) L.S. $= \dfrac{\cos\theta - \cot\theta}{1 - \sin\theta}$

$= \dfrac{\cos\theta - \frac{\cos\theta}{\sin\theta}}{1 - \sin\theta}$

$= \dfrac{(\cos\theta)(\sin\theta) - \cos\theta}{(1 - \sin\theta)(\sin\theta)}$

$= \dfrac{(\cos\theta)(\sin\theta - 1)}{(1 - \sin\theta)(\sin\theta)}$

$= \dfrac{-\cos\theta}{\sin\theta}$

$= -\cot\theta$

$=$ R.S.

The left side is equal to the right side, so the identity is proved.

10. Use algebra to solve each equation over the domain $0 \le x < 2\pi$. Give the roots to the nearest hundredth where necessary.

a) $\tan x = \cot x$

Assume $\tan x \neq 0$, then divide by $\tan x$.

$\frac{\tan x}{\tan x} = \frac{\cot x}{\tan x}$

$1 = \frac{1}{\tan^2 x}$

$\tan^2 x = 1$

$\tan x = \pm 1$

$x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}$, or

$x = \frac{7\pi}{4}$

For $\tan x = 0$, $x = 0$ or $x = \pi$

Verify by substitution that neither value of x is a root of the given equation.

Verify the roots by substitution.

b) $\cos x + \sqrt{3} \sin x = 0$

Assume $\cos x \neq 0$, then divide by $\cos x$.

$\frac{\cos x}{\cos x} + \sqrt{3}\frac{\sin x}{\cos x} = 0$

$1 + \sqrt{3} \tan x = 0$

$\tan x = -\frac{1}{\sqrt{3}}$

$x = \frac{5\pi}{6}$ or $x = \frac{11\pi}{6}$

For $\cos x = 0$, $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

Verify by substitution that neither value of x is a root of the given equation.

Verify the roots by substitution.

c) $2 \cos x = 7 - 3 \sec x$

$2 \cos x = 7 - \frac{3}{\cos x}$

Multiply by $\cos x$, then collect terms on one side.

$2 \cos^2 x - 7 \cos x + 3 = 0$

$(2 \cos x - 1)(\cos x - 3) = 0$

Either $2 \cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

Or $\cos x - 3 = 0$

$\cos x = 3$

This equation has no solution.

Verify the roots by substitution.

d) $\sin^2 x = \sin x \cos x$

$\sin^2 x - \sin x \cos x = 0$

$(\sin x)(\sin x - \cos x) = 0$

Either $\sin x = 0$

$x = 0$ or $x = \pi$

Or $\sin x - \cos x = 0$

$\sin x = \cos x$

Assume $\cos x \neq 0$, then divide by $\cos x$.

$\tan x = 1$

$x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$

For $\cos x = 0$, $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

Verify by substitution that neither value of x is a root of the given equation.

Verify the roots by substitution.

11. Identify any errors in this proof, then write a correct algebraic proof.

To prove: $\frac{\sin\theta}{1 - \sin\theta} = \frac{1}{\csc\theta - 1}$

$$\begin{aligned}
\text{L.S.} &= \frac{\sin\theta}{1 - \sin\theta} \\
&= \frac{\sin\theta}{1} - \frac{\sin\theta}{\sin\theta} \\
&= \sin\theta - 1 \\
&= \frac{1}{\csc\theta} - 1 \\
&= \frac{1}{\csc\theta - 1} \\
&= \text{R.S.}
\end{aligned}$$

Correct proof:

$$\begin{aligned}
\text{R.S.} &= \frac{1}{\csc\theta - 1} \\
&= \frac{1}{\frac{1}{\sin\theta} - 1} \\
&= \frac{1}{\frac{1 - \sin\theta}{\sin\theta}} \\
&= \frac{\sin\theta}{1 - \sin\theta} \\
&= \text{L.S.}
\end{aligned}$$

From the 1st line of the proof to the 2nd line, $\frac{\sin\theta}{1 - \sin\theta}$ cannot be written as $\frac{\sin\theta}{1} - \frac{\sin\theta}{\sin\theta}$.

From the 4th line of the proof to the 5th line, $\frac{1}{\csc\theta} - 1$ cannot be written as $\frac{1}{\csc\theta - 1}$.

12. Identify which equation below is an identity. Justify your answer. Prove the identity. Solve the other equation over the domain $-\pi \leq x \leq \pi$. Give the roots to the nearest hundredth.

a) $\frac{2 \sin^2 x + 1}{\sin x} = 2 \csc^2 x - 1$

I graphed each side of the equation. The graphs do not coincide, so this is an equation.
From the graphs, the roots are approximately: $x = 0.81$ and $x = 2.33$

b) $\frac{\sin^2 x + 1}{\sin x} = \frac{1 + \csc^2 x}{\csc x}$

I graphed each side of the equation. The graphs appear to coincide, so this is probably the identity.

$$\text{R.S.} = \frac{1 + \csc^2 x}{\csc x}$$

$$= \frac{1 + \frac{1}{\sin^2 x}}{\frac{1}{\sin x}}$$

Multiply numerator and denominator by $\sin^2 x$.

$$\text{R.S.} = \frac{\sin^2 x + 1}{\sin x}$$

$$= \text{L.S.}$$

The left side is equal to the right side, so the identity is proved.

C

13. Here are two identities that involve the cotangent ratio:
$\cot \theta = \frac{1}{\tan \theta}$ and $\cot \theta = \frac{\cos \theta}{\sin \theta}$

a) Show how you can derive one identity from the other.

$\cot \theta = \frac{1}{\tan \theta}$ Write $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

$\cot \theta = \frac{1}{\frac{\sin \theta}{\cos \theta}}$ Multiply numerator and denominator by $\cos \theta$.

$\cot \theta = \frac{\cos \theta}{\sin \theta}$

b) Determine the non-permissible values of θ for each identity. Explain why these values are different. How could you illustrate this using graphing technology?

For $\cot \theta = \frac{1}{\tan \theta}$

Since $\tan \theta = \frac{\sin \theta}{\cos \theta}$, then

$\sin \theta \neq 0$ and $\cos \theta \neq 0$

So, $\theta \neq \pi k, k \in \mathbb{Z}$ and $\theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

For $\cot \theta = \frac{\cos \theta}{\sin \theta}$

$\sin \theta \neq 0$

So, $\theta \neq \pi k, k \in \mathbb{Z}$

The values are different because there are two restrictions for $\frac{1}{\tan \theta}$ and only one restriction for $\frac{\cos \theta}{\sin \theta}$.

When I graph $y = \frac{1}{\tan \theta}$, and set the TABLE for intervals of $\frac{\pi}{2}$, it shows ERROR for all values of X in the table. When I graph $y = \frac{\cos \theta}{\sin \theta}$, with the same TABLE settings, it shows ERROR only for $X = \pi k, k \in \mathbb{Z}$.