Lesson 7.4 Exercises, pages 626–632

A

3. For each expression below:

i) Determine any non-permissible values of θ.

ii) Write the expression as a single term.

a) $1 - \cos^2\theta$

i) All real values are permissible.

ii) $1 - \cos^2\theta = \sin^2\theta$

b) $\cos^2\theta - 1$

i) All real values are permissible.

ii) $\cos^2\theta - 1 = -\sin^2\theta$

c) $\sec^2\theta - 1$

i) $\sec^2\theta = \frac{1}{\cos^2\theta}$, so $\cos\theta \neq 0$, $\theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

ii) $\sec^2\theta - 1 = \tan^2\theta$

d) $1 - \sec^2\theta$

i) $\sec^2\theta = \frac{1}{\cos^2\theta}$, so $\cos\theta \neq 0$, $\theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

ii) $1 - \sec^2\theta = -\tan^2\theta$

e) $\csc^2\theta - \cot^2\theta$

i) $\csc^2\theta = \frac{1}{\sin^2\theta}$ and $\cot^2\theta = \frac{\cos^2\theta}{\sin^2\theta}$, so $\sin\theta \neq 0$, $\theta \neq \pi k, k \in \mathbb{Z}$

ii) $\csc^2\theta - \cot^2\theta = 1$

f) $\sin^2\theta + \cos^2\theta + 1$

i) All real values are permissible.

ii) $\sin^2\theta + \cos^2\theta + 1 = 1 + 1$
$= 2$

4. a) Verify the identity $\tan^2\theta + 1 = \sec^2\theta$ for $\theta = \frac{2\pi}{3}$.

L.S. $= \tan^2\theta + 1$
$= \tan^2\left(\frac{2\pi}{3}\right) + 1$
$= (-\sqrt{3})^2 + 1$
$= 4$

R.S. $= \sec^2\theta$
$= \sec^2\left(\frac{2\pi}{3}\right)$
$= (-2)^2$
$= 4$

The left side is equal to the right side, so the identity is verified.

b) Verify the identity $1 + \cot^2\theta = \csc^2\theta$ using graphing technology.

The graphs of $y = 1 + \frac{1}{\tan^2\theta}$ and $y = \frac{1}{\sin^2\theta}$ coincide, so the identity is verified.

c) Verify the identity $\sin^2\theta + \cos^2\theta = 1$ for $\theta = 300°$.

$$\begin{aligned}\text{L.S.} &= \sin^2\theta + \cos^2\theta \\ &= \sin^2(300°) + \cos^2(300°) \\ &= \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ &= 1 \\ &= \text{R.S.}\end{aligned}$$

The left side is equal to the right side, so the identity is verified.

B

5. For each expression below:

i) Determine any non-permissible values of θ.

ii) Write the expression as a single term.

a) $\dfrac{\sqrt{1 - \sin^2\theta}}{\sqrt{1 + \tan^2\theta}}$

i) $\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$, so $\cos\theta \neq 0$, $\theta \neq \dfrac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$

ii) $$\begin{aligned}&\frac{\sqrt{1 - \sin^2\theta}}{\sqrt{1 + \tan^2\theta}} \\ &= \frac{\sqrt{\cos^2\theta}}{\sqrt{\sec^2\theta}} \\ &= \frac{|\cos\theta|}{\frac{1}{|\cos\theta|}} \\ &= |\cos\theta|^2 \\ &= \cos^2\theta\end{aligned}$$

b) $\dfrac{1 - \sin^2\theta + \cos^2\theta}{\cos\theta}$

i) $\cos\theta \neq 0$, so $\theta \neq \dfrac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$

ii) $$\begin{aligned}&\frac{1 - \sin^2\theta + \cos^2\theta}{\cos\theta} \\ &= \frac{\cos^2\theta + \cos^2\theta}{\cos\theta} \\ &= \frac{2\cos^2\theta}{\cos\theta} \\ &= 2\cos\theta\end{aligned}$$

c) $\dfrac{\cos\theta}{1 + \sin\theta} + \dfrac{\cos\theta}{1 - \sin\theta}$

i) $\sin\theta \neq \pm 1$, so $\theta \neq \dfrac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$

ii) $$\begin{aligned}&\frac{\cos\theta}{1 + \sin\theta} + \frac{\cos\theta}{1 - \sin\theta} \\ &= \frac{\cos\theta(1 - \sin\theta)}{(1 + \sin\theta)(1 - \sin\theta)} + \frac{\cos\theta(1 + \sin\theta)}{(1 - \sin\theta)(1 + \sin\theta)} \\ &= \frac{\cos\theta - \cos\theta\sin\theta + \cos\theta + \cos\theta\sin\theta}{1 - \sin^2\theta} \\ &= \frac{2\cos\theta}{\cos^2\theta} \\ &= \frac{2}{\cos\theta} \\ &= 2\sec\theta\end{aligned}$$

d) $\dfrac{\csc\theta}{\cot\theta + \tan\theta}$

i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, so $\cos\theta \neq 0$, $\theta \neq \dfrac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$; $\csc\theta = \dfrac{1}{\sin\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, so $\sin\theta \neq 0$, $\theta \neq \pi k$, $k \in \mathbb{Z}$

ii) $$\begin{aligned}&\frac{\csc\theta}{\cot\theta + \tan\theta} \\ &= \frac{\frac{1}{\sin\theta}}{\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}}\end{aligned}$$

Multiply numerator and denominator by $\cos\theta\sin\theta$.

$$\begin{aligned}&= \frac{\cos\theta}{\cos^2\theta + \sin^2\theta} \\ &= \frac{\cos\theta}{1}, \text{ or } \cos\theta\end{aligned}$$

6. For each identity:

i) Verify the identity using graphing technology.

ii) Prove the identity.

a) $1 - \cos^2\theta = \cos^2\theta \tan^2\theta$

i) The graphs of $y = 1 - \cos^2\theta$ and $y = \cos^2\theta \tan^2\theta$ coincide, so the identity is verified.

ii) $$\begin{aligned} \text{R.S.} &= \cos^2\theta \tan^2\theta \\ &= \cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta}\right) \\ &= \sin^2\theta \\ &= 1 - \cos^2\theta \\ &= \text{L.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

b) $\sin^2\theta + \cos^2\theta + \tan^2\theta = \sec^2\theta$

i) The graphs of $y = \dfrac{1}{\cos^2\theta}$ and $y = \sin^2\theta + \cos^2\theta + \tan^2\theta$ coincide, so the identity is verified.

ii) $$\begin{aligned} \text{L.S.} &= \sin^2\theta + \cos^2\theta + \tan^2\theta \\ &= 1 + \tan^2\theta \\ &= \sec^2\theta \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

7. For each identity:

i) Verify the identity for $\theta = 240°$. **ii)** Prove the identity.

a) $\cot^2\theta \sec\theta + \dfrac{1}{\cos\theta} = \csc^2\theta \sec\theta$

i) Substitute: $\theta = 240°$

$$\begin{aligned} \text{L.S.} &= \cot^2\theta \sec\theta + \frac{1}{\cos\theta} \\ &= (\cot 240°)^2(\sec 240°) + \frac{1}{\cos 240°} \\ &= \left(\frac{1}{\sqrt{3}}\right)^2(-2) + (-2) \\ &= -\frac{8}{3} \end{aligned}$$

$$\begin{aligned} \text{R.S.} &= \csc^2\theta \sec\theta \\ &= (\csc 240°)^2(\sec 240°) \\ &= \left(-\frac{2}{\sqrt{3}}\right)^2(-2) \\ &= -\frac{8}{3} \end{aligned}$$

The left side is equal to the right side, so the identity is verified.

ii) $$\begin{aligned} \text{L.S.} &= \cot^2\theta \sec\theta + \frac{1}{\cos\theta} \\ &= \cot^2\theta \sec\theta + \sec\theta \\ &= (\sec\theta)(\cot^2\theta + 1) \\ &= (\sec\theta)(\csc^2\theta) \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

b) $\dfrac{\tan\theta}{\cos\theta - \sec\theta} = -\csc\theta$

i) Substitute: $\theta = 240°$

$$\begin{aligned} \text{L.S.} &= \frac{\tan\theta}{\cos\theta - \sec\theta} \\ &= \frac{\tan 240°}{\cos 240° - \sec 240°} \\ &= \frac{\sqrt{3}}{\left(-\frac{1}{2}\right) - (-2)} \\ &= \frac{\sqrt{3}}{\frac{3}{2}}, \text{ or } \frac{2}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \text{R.S.} &= -\csc\theta \\ &= -\csc 240° \\ &= -\left(-\frac{2}{\sqrt{3}}\right), \text{ or } \frac{2}{\sqrt{3}} \end{aligned}$$

The left side is equal to the right side, so the identity is verified.

ii) $$\begin{aligned} \text{L.S.} &= \frac{\tan\theta}{\cos\theta - \sec\theta} \\ &= \frac{\frac{\sin\theta}{\cos\theta}}{\cos\theta - \frac{1}{\cos\theta}} \\ &= \frac{\sin\theta}{\cos^2\theta - 1} \\ &= \frac{\sin\theta}{-\sin^2\theta} \\ &= \frac{1}{-\sin\theta} \\ &= -\csc\theta \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

8. Is either of these statements true? Justify your answer.

a) Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, then $\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$

This statement is true because if $c = \frac{a}{b}$, then I can square both sides to obtain $c^2 = \frac{a^2}{b^2}$.

b) Since $\sin^2\theta + \cos^2\theta = 1$, then $\sin\theta + \cos\theta = 1$

This statement is false. For example, for $\theta = \frac{\pi}{4}$, $\sin^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right) = 1$, but $\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$, or $\frac{2}{\sqrt{2}}$, which is not 1.

9. For each identity:

i) Determine the non-permissible values of θ.

ii) Prove the identity.

a) $\frac{1}{\csc\theta + \cot\theta} = \csc\theta - \cot\theta$

i) $\csc\theta = \frac{1}{\sin\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$, so $\sin\theta \neq 0$,
$\theta \neq \pi k, k \in \mathbb{Z}$
$\csc\theta + \cot\theta \neq 0$,
so $\frac{1}{\sin\theta} \neq -\frac{\cos\theta}{\sin\theta}$,
$\cos\theta \neq -1$, and
$\theta \neq (2k + 1)\pi, k \in \mathbb{Z}$

ii)
$$\begin{aligned}
\text{R.S.} &= \csc\theta - \cot\theta \\
&= \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} \\
&= \frac{1 - \cos\theta}{\sin\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta} \\
&= \frac{1 - \cos^2\theta}{(\sin\theta)(1 + \cos\theta)} \\
&= \frac{\sin^2\theta}{\sin\theta + \sin\theta\cos\theta} \\
&= \frac{\frac{\sin^2\theta}{\sin^2\theta}}{\frac{\sin\theta}{\sin^2\theta} + \frac{\sin\theta\cos\theta}{\sin^2\theta}} \\
&= \frac{1}{\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}} \\
&= \frac{1}{\csc\theta + \cot\theta} \\
&= \text{L.S.}
\end{aligned}$$

The left side is equal to the right side, so the identity is proved.

b) $\sin\theta + \frac{\cos\theta}{\tan\theta} = \frac{1}{\cos\theta\tan\theta}$

i) $\tan\theta = \frac{\sin\theta}{\cos\theta}$, so $\cos\theta \neq 0$,
$\theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$ and
$\sin\theta \neq 0, \theta \neq \pi k, k \in \mathbb{Z}$

ii)
$$\begin{aligned}
\text{L.S.} &= \sin\theta + \frac{\cos\theta}{\tan\theta} \\
&= \sin\theta + \frac{\cos\theta}{\frac{\sin\theta}{\cos\theta}} \\
&= \sin\theta + \frac{\cos^2\theta}{\sin\theta} \\
&= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta} \\
&= \frac{1}{\sin\theta} \\
&= \frac{1}{(\sin\theta)\left(\frac{\cos\theta}{\cos\theta}\right)} \\
&= \frac{1}{(\cos\theta)\left(\frac{\sin\theta}{\cos\theta}\right)} \\
&= \frac{1}{\cos\theta\tan\theta} \\
&= \text{R.S.}
\end{aligned}$$

The left side is equal to the right side, so the identity is proved.

10. Use algebra to solve each equation over the domain $-90° \leq x \leq 270°$. Give the roots to the nearest degree.

a) $4 - 4\cos^2 x = \sin x$

$4 - 4(1 - \sin^2 x) - \sin x = 0$
$4 - 4 + 4\sin^2 x - \sin x = 0$
$4\sin^2 x - \sin x = 0$
$(\sin x)(4\sin x - 1) = 0$
Either $\sin x = 0$
$x = 0°, x = 180°$
Or $4\sin x - 1 = 0$
$4\sin x = 1$
$\sin x = \frac{1}{4}$
$x \doteq 14°, x \doteq 166°$
The roots are: $x = 0°$, $x = 180°, x \doteq 14°, x \doteq 166°$
Verify by substitution.

b) $\cos x + 1 = 2\sin^2 x$

$\cos x + 1 - 2(1 - \cos^2 x) = 0$
$\cos x + 1 - 2 + 2\cos^2 x = 0$
$2\cos^2 x + \cos x - 1 = 0$
$(2\cos x - 1)(\cos x + 1) = 0$
Either $2\cos x - 1 = 0$
$2\cos x = 1$
$\cos x = \frac{1}{2}$
$x = 60°, x = -60°$
Or $\cos x + 1 = 0$
$\cos x = -1$
$x = 180°$
The roots are: $x = 60°, x = -60°$, $x = 180°$
Verify by substitution.

11. Identify whether each equation is an identity. Justify your answer. Prove each identity. Use algebra to solve each equation that is not an identity over the domain $-\pi \leq x \leq \pi$. Give the roots to the nearest hundredth.

a) $\cos^2 x = (\sin x)(\csc x + \sin x)$

The graphs of $y = \cos^2 x$ and $y = (\sin x)\left(\frac{1}{\sin x} + \sin x\right)$ do not coincide, so the equation is not an identity.

$\cos^2 x = (\sin x)\left(\frac{1}{\sin x} + \sin x\right), \sin x \neq 0$
$\cos^2 x = 1 + \sin^2 x$ Substitute: $\cos^2 x = 1 - \sin^2 x$
$1 - \sin^2 x = 1 + \sin^2 x$
$2\sin^2 x = 0$
$\sin^2 x = 0$
$\sin x = 0$
Since this is a non-permissible value, the equation has no real solution.

b) $(\cos x)(\sec x - \cos x) = \cos^2 x$

The graphs of $y = (\cos x)\left(\frac{1}{\cos x} - \cos x\right)$ and $y = \cos^2 x$ do not coincide, so the equation is not an identity.
$(\cos x)(\sec x) - \cos^2 x = \cos^2 x, \cos x \neq 0$
$1 - 2\cos^2 x = 0$
$\cos^2 x = \frac{1}{2}$
$\cos x = \pm\sqrt{\frac{1}{2}}$
The roots are: $x \doteq 0.79, x \doteq 2.36, x \doteq -2.36, x \doteq -0.79$
Verify by substitution.

12. a) Prove this identity: $\frac{\cot\theta}{\csc\theta + 1} = \frac{\csc\theta - 1}{\cot\theta}$

$$\begin{aligned}\text{L.S.} &= \frac{\cot\theta}{\csc\theta + 1}\\ &= \frac{\cot\theta}{(\csc\theta + 1)} \cdot \frac{(\csc\theta - 1)}{(\csc\theta - 1)}\\ &= \frac{(\cot\theta)(\csc\theta - 1)}{\csc^2\theta - 1}\\ &= \frac{(\cot\theta)(\csc\theta - 1)}{\cot^2\theta}\\ &= \frac{\csc\theta - 1}{\cot\theta}\\ &= \text{R.S.}\end{aligned}$$

Multiply numerator and denominator by the conjugate of the denominator.

Since the left side is equal to the right side, the identity is proved.

b) Predict a similar identity involving tan θ and sec θ. Prove this identity.

A similar identity is: $\frac{\tan\theta}{\sec\theta + 1} = \frac{\sec\theta - 1}{\tan\theta}$

$$\begin{aligned}\text{L.S.} &= \frac{\tan\theta}{\sec\theta + 1}\\ &= \frac{\tan\theta}{(\sec\theta + 1)} \cdot \frac{(\sec\theta - 1)}{(\sec\theta - 1)}\\ &= \frac{(\tan\theta)(\sec\theta - 1)}{(\sec^2\theta - 1)}\\ &= \frac{(\tan\theta)(\sec\theta - 1)}{\tan^2\theta}\\ &= \frac{\sec\theta - 1}{\tan\theta}\\ &= \text{R.S.}\end{aligned}$$

Multiply numerator and denominator by the conjugate of the denominator.

Since the left side is equal to the right side, the identity is proved.

C

13. Determine a single trigonometric function for m such that the equation $\frac{2 - \sin^2\theta}{\cos\theta} = m + \cos\theta$ is an identity. Verify your answer by proving the identity.

$$\frac{2 - \sin^2\theta}{\cos\theta} = m + \cos\theta$$ Solve for m.

$$m = \frac{2 - \sin^2\theta}{\cos\theta} - \cos\theta$$ Use a common denominator.

$$= \frac{2 - \sin^2\theta - \cos^2\theta}{\cos\theta}$$

$$= \frac{2 - (\sin^2\theta + \cos^2\theta)}{\cos\theta}$$

$$= \frac{2 - 1}{\cos\theta}$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta$$

The identity is: $\frac{2 - \sin^2\theta}{\cos\theta} = \sec\theta + \cos\theta$

$$\text{L.S.} = \frac{2 - \sin^2\theta}{\cos\theta}$$

$$= \frac{2 - (1 - \cos^2\theta)}{\cos\theta}$$

$$= \frac{1 + \cos^2\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\cos^2\theta}{\cos\theta}$$

$$= \sec\theta + \cos\theta$$

$$= \text{R.S.}$$

Since the left side is equal to the right side, the identity is proved.