Lesson 7.6 Exercises, pages 658–665

A

4. Write each expression as a single trigonometric ratio.

a) $\sin(\theta + \theta)$

$= \sin 2\theta$

b) $\sin\theta + \sin\theta$

$= 2\sin\theta$

c) $\sin\theta\sin\theta$

$= \sin^2\theta$

d) $\cos^2\theta\cos\theta$

$= \cos^3\theta$

e) $3\sin\theta\sin\theta$

$= 3\sin^2\theta$

f) $3\sin\theta + \sin\theta$

$= 4\sin\theta$

5. Determine the exact value of each expression.

a) $2\sin 45° \cos 45°$

$= \sin 2(45°)$
$= \sin 90°$
$= 1$

b) $\cos^2\left(\frac{\pi}{6}\right)\sin^2\left(\frac{\pi}{6}\right)$

$= \left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right)^2$
$= \frac{3}{16}$

c) $2\cos^2\left(\frac{\pi}{8}\right) - 1$

$= \cos 2\left(\frac{\pi}{8}\right)$
$= \cos\frac{\pi}{4}$
$= \frac{1}{\sqrt{2}}$

d) $\frac{2\tan 210°}{1 - \tan^2(210°)}$

$= \tan 2(210°)$
$= \tan 420°$
$= \tan 60°$
$= \sqrt{3}$

6. Simplify each expression.

a) $-\cos^2\theta - \sin^2\theta$

$= -(\cos^2\theta + \sin^2\theta)$
$= -1$

b) $1 - \sin^2\theta$

$= \cos^2\theta$

c) $2\cos^2\theta - 2$

$= 2(\cos^2\theta - 1)$
$= -2\sin^2\theta$

d) $4\cos^2\theta - 2$

$= 2(2\cos^2\theta - 1)$
$= 2\cos 2\theta$

B

7. Verify each identity for the given value of θ.

a) $\sin 2\theta = 2\sin\theta\cos\theta; \theta = 30°$

$$\begin{aligned}\text{L.S.} &= \sin 2\theta\\ &= \sin 60°\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\text{R.S.} &= 2\sin\theta\cos\theta\\ &= 2\sin 30°\cos 30°\\ &= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

The left side is equal to the right side, so the identity is verified.

b) $\cos 2\theta = \cos^2\theta - \sin^2\theta; \theta = \frac{5\pi}{4}$

$$\begin{aligned}\text{L.S.} &= \cos 2\theta\\ &= \cos\frac{10\pi}{4}\\ &= \cos\frac{\pi}{2}\\ &= 0\end{aligned}$$

$$\begin{aligned}\text{R.S.} &= \cos^2\theta - \sin^2\theta\\ &= \left(\cos\frac{5\pi}{4}\right)^2 - \left(\sin\frac{5\pi}{4}\right)^2\\ &= \left(-\frac{1}{\sqrt{2}}\right)^2 - \left(-\frac{1}{\sqrt{2}}\right)^2\\ &= 0\end{aligned}$$

The left side is equal to the right side, so the identity is verified.

8. For the identity $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$, determine the non-permissible values of θ over the set of real numbers, then verify the identity for $\theta = \frac{2\pi}{3}$.

$\cos 2\theta \neq 0$
$2\theta \neq \frac{\pi}{2} + \pi k,$
$k \in \mathbb{Z}$
$\theta \neq \frac{\pi}{4} + \frac{\pi}{2}k,$
$k \in \mathbb{Z}$

$\cos\theta \neq 0$
$\theta \neq \frac{\pi}{2} + \pi k,$
$k \in \mathbb{Z}$

$\tan^2\theta \neq 1$
$\tan\theta \neq \pm 1$
$\theta \neq \frac{\pi}{4}(2k + 1), k \in \mathbb{Z}$

To verify, substitute $\theta = \frac{2\pi}{3}$ in each side of the identity.

$$\begin{aligned}\text{L.S.} &= \tan 2\theta\\ &= \tan\frac{4\pi}{3}\\ &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}\text{R.S.} &= \frac{2\tan\theta}{1 - \tan^2\theta}\\ &= \frac{2\tan\frac{2\pi}{3}}{1 - \tan^2\left(\frac{2\pi}{3}\right)}\\ &= \frac{2(-\sqrt{3})}{1 - (-\sqrt{3})^2}\\ &= \sqrt{3}\end{aligned}$$

The left side is equal to the right side, so the identity is verified.

9. Given angle θ is in standard position with its terminal arm in Quadrant 2 and $\tan\theta = -\frac{2}{5}$, determine the exact value of each trigonometric ratio.

a) $\sin 2\theta$ **b)** $\cos 2\theta$ **c)** $\tan 2\theta$

Since the terminal arm of θ lies in Quadrant 2, $\sin\theta$ is positive.
Use mental math and the Pythagorean Theorem to determine the value of r, which is $\sqrt{29}$.

So, $\sin\theta = \frac{2}{\sqrt{29}}$ and $\cos\theta = -\frac{5}{\sqrt{29}}$

a) Substitute values of $\sin\theta$ and $\cos\theta$ in:

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{2}{\sqrt{29}}\right)\left(-\frac{5}{\sqrt{29}}\right)\\ &= -\frac{20}{29}\end{aligned}$$

b) Substitute the value of $\cos\theta$ in:

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{5}{\sqrt{29}}\right)^2 - 1\\ &= \frac{50}{29} - 1, \text{ or } \frac{21}{29}\end{aligned}$$

c) Substitute $\tan\theta = -\frac{2}{5}$ in $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$:

$$\begin{aligned}\tan 2\theta &= \frac{2\left(-\frac{2}{5}\right)}{1 - \left(-\frac{2}{5}\right)^2}\\ &= \frac{-\frac{4}{5}}{\frac{21}{25}}, \text{ or } -\frac{20}{21}\end{aligned}$$

10. Prove each identity.

a) $2\cot\theta\sin^2\theta = \sin 2\theta$

$$\begin{aligned}\text{L.S.} &= 2\cot\theta\sin^2\theta\\ &= 2\left(\frac{\cos\theta}{\sin\theta}\right)\sin^2\theta\\ &= 2\cos\theta\sin\theta\\ &= \sin 2\theta\\ &= \text{R.S.}\end{aligned}$$

The left side is equal to the right side, so the identity is proved.

b) $1 + \sin 2\theta = (\sin\theta + \cos\theta)^2$

$$\begin{aligned}\text{R.S.} &= (\sin\theta + \cos\theta)^2\\ &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\\ &= 1 + \sin 2\theta\\ &= \text{L.S.}\end{aligned}$$

The left side is equal to the right side, so the identity is proved.

c) $2\cot\theta = \frac{\sin 2\theta}{1 - \cos^2\theta}$

$$\begin{aligned}\text{R.S.} &= \frac{\sin 2\theta}{1 - \cos^2\theta}\\ &= \frac{2\sin\theta\cos\theta}{\sin^2\theta}\\ &= \frac{2\cos\theta}{\sin\theta}\\ &= 2\cot\theta\\ &= \text{L.S.}\end{aligned}$$

The left side is equal to the right side, so the identity is proved.

d) $\tan 2\theta\cos 2\theta = 2\sin\theta\cos\theta$

$$\begin{aligned}\text{L.S.} &= \tan 2\theta\cos 2\theta\\ &= \left(\frac{\sin 2\theta}{\cos 2\theta}\right)\cos 2\theta\\ &= \sin 2\theta\\ &= 2\sin\theta\cos\theta\\ &= \text{R.S.}\end{aligned}$$

The left side is equal to the right side, so the identity is proved.

11. Prove each identity.

a) $\dfrac{\cos 2\theta - 2\cos^2\theta}{2\cos\theta} = \dfrac{-\sin 2\theta \sec^2\theta}{4\sin\theta}$

$$\text{L.S.} = \frac{\cos 2\theta - 2\cos^2\theta}{2\cos\theta}$$
$$= \frac{2\cos^2\theta - 1 - 2\cos^2\theta}{2\cos\theta}$$
$$= \frac{-1}{2\cos\theta}$$
$$\text{R.S.} = \frac{-\sin 2\theta \sec^2\theta}{4\sin\theta}$$
$$= \frac{\frac{-2\sin\theta\cos\theta}{\cos^2\theta}}{4\sin\theta}$$
$$= \frac{\frac{-1}{\cos\theta}}{2}$$
$$= \frac{-1}{2\cos\theta}$$

The left side and the right side simplify to the same expression, so the identity is proved.

b) $\dfrac{\sin 2\theta}{\cos 2\theta - 1} = -\cot\theta$

$$\text{L.S.} = \frac{\sin 2\theta}{\cos 2\theta - 1}$$
$$= \frac{2\sin\theta\cos\theta}{1 - 2\sin^2\theta - 1}$$
$$= \frac{2\sin\theta\cos\theta}{-2\sin^2\theta}$$
$$= -\frac{\cos\theta}{\sin\theta}$$
$$= -\cot\theta$$
$$= \text{R.S.}$$

The left side is equal to the right side, so the identity is proved.

c) $\dfrac{2\cos^2\theta}{\sin 2\theta} = \dfrac{1 + \cos\theta}{\tan\theta + \sin\theta}$

$$\text{L.S.} = \frac{2\cos^2\theta}{\sin 2\theta}$$
$$= \frac{2\cos\theta\cos\theta}{2\sin\theta\cos\theta}$$
$$= \frac{\cos\theta}{\sin\theta}$$
$$\text{R.S.} = \frac{1 + \cos\theta}{\tan\theta + \sin\theta}$$
$$= \frac{1 + \cos\theta}{\frac{\sin\theta}{\cos\theta} + \sin\theta}$$
$$= \frac{(1 + \cos\theta)(\cos\theta)}{\sin\theta + \sin\theta\cos\theta}$$
$$= \frac{(1 + \cos\theta)(\cos\theta)}{(\sin\theta)(1 + \cos\theta)}$$
$$= \frac{\cos\theta}{\sin\theta}$$

The left side and the right side simplify to the same expression, so the identity is proved.

d) $\dfrac{\sin 2\theta}{2 + 2\cos 2\theta} = \dfrac{\sec^2\theta - 1}{2\tan\theta}$

$$\text{L.S.} = \frac{\sin 2\theta}{2 + 2\cos 2\theta}$$
$$= \frac{2\sin\theta\cos\theta}{2(1 + \cos 2\theta)}$$
$$= \frac{\sin\theta\cos\theta}{1 + 2\cos^2\theta - 1}$$
$$= \frac{\sin\theta\cos\theta}{2\cos\theta\cos\theta}$$
$$= \frac{\sin\theta}{2\cos\theta}$$
$$= \frac{1}{2}\tan\theta$$
$$\text{R.S.} = \frac{\sec^2\theta - 1}{2\tan\theta}$$
$$= \frac{\tan^2\theta}{2\tan\theta}$$
$$= \frac{1}{2}\tan\theta$$

The left side and right side simplify to the same expression, so the identity is proved.

12. Solve each equation over the domain $-180° \leq x < 180°$.

a) $\cos 2x = 2\cos x - 1$

$2\cos^2 x - 1 - 2\cos x + 1 = 0$
$2\cos^2 x - 2\cos x = 0$
$(2\cos x)(\cos x - 1) = 0$
Either $2\cos x = 0$
$\cos x = 0$
$x = 90°$ or $x = -90°$
Or $\cos x - 1 = 0$
$\cos x = 1$
$x = 0°$
The roots are: $x = 0°$, $x = 90°$, and $x = -90°$

b) $\sin x = \cos 2x$

$\sin x = 1 - 2\sin^2 x$
$2\sin^2 x + \sin x - 1 = 0$
$(2\sin x - 1)(\sin x + 1) = 0$
Either $2\sin x - 1 = 0$
$\sin x = \frac{1}{2}$
$x = 30°$ or $x = 150°$
Or $\sin x + 1 = 0$
$\sin x = -1$
$x = -90°$
The roots are: $x = 30°$, $x = 150°$, and $x = -90°$

13. Solve each equation over the domain $-2\pi < x < 2\pi$.

a) $\sqrt{2}\sin 2x - 2\sin x = 0$

$\sqrt{2}(2\sin x\cos x) - 2\sin x = 0$
$2\sin x(\sqrt{2}\cos x - 1) = 0$
Either $2\sin x = 0$
$\sin x = 0$
$x = 0$ or $x = \pi$ or $x = -\pi$
Or $\sqrt{2}\cos x - 1 = 0$
$\cos x = \frac{1}{\sqrt{2}}$
$x = \frac{\pi}{4}$ or $x = \frac{7\pi}{4}$ or $x = -\frac{\pi}{4}$ or
$x = -\frac{7\pi}{4}$
The roots are: $x = 0$, $x = \pm\frac{\pi}{4}$, $x = \pm\pi$, and $x = \pm\frac{7\pi}{4}$

b) $3\sin^2 x + \cos 2x - 2 = 0$

$3\sin^2 x + 1 - 2\sin^2 x - 2 = 0$
$\sin^2 x - 1 = 0$
$\sin x = \pm 1$
$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$ or $x = -\frac{\pi}{2}$ or
$x = -\frac{3\pi}{2}$
The roots are: $x = \pm\frac{\pi}{2}$ and $x = \pm\frac{3\pi}{2}$

14. For this solution of the equation $\sin 2x = \cos x$ over the domain $0 \leq x < 2\pi$, identify the error then write a correct solution.

$\sin 2x = \cos x$
$2\sin x\cos x = \cos x$
$\frac{2\sin x \cancel{\cos x}^{1}}{\cancel{\cos x}_{1}} = \frac{\cancel{\cos x}^{1}}{\cancel{\cos x}_{1}}$
$2\sin x = 1$
$\sin x = \frac{1}{2}$
$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

$\sin 2x = \cos x$
$2\sin x\cos x - \cos x = 0$
$\cos x(2\sin x - 1) = 0$
Either $\cos x = 0$
$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$
Or $2\sin x = 1$
$\sin x = \frac{1}{2}$
$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$
The roots are: $x = \frac{\pi}{6}$, $x = \frac{\pi}{2}$, $x = \frac{5\pi}{6}$, and $x = \frac{3\pi}{2}$

In the 3rd line, $\cos x$ cannot be removed, because a possible solution is $\cos x = 0$.

15. Use algebra to solve the equation cos $2x = 2\cos x$ over the set of real numbers. Give the answer to the nearest hundredth.

$$\cos 2x = 2\cos x$$
$$2\cos^2 x - 1 = 2\cos x$$
$$2\cos^2 x - 2\cos x - 1 = 0$$

Use: $\cos x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Substitute: $a = 2, b = -2, c = -1$

$$\cos x = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$

$$\cos x = \frac{2 \pm \sqrt{12}}{4}$$

$$\cos x = \frac{2 \pm 2\sqrt{3}}{4}$$

$$\cos x = \frac{1 \pm \sqrt{3}}{2}$$

$\dfrac{1 + \sqrt{3}}{2} > 1$, so there is no solution for $\cos x = \dfrac{1 + \sqrt{3}}{2}$.

So, $\cos x = \dfrac{1 - \sqrt{3}}{2}$

Since $\dfrac{1 - \sqrt{3}}{2}$ is negative, the terminal arm of angle x lies in Quadrant 2 or 3.

The reference angle is: $\cos^{-1}\left(\dfrac{\sqrt{3} - 1}{2}\right) = 1.1960...$

In Quadrant 2, $x = \pi - 1.1960...$, or $1.9455...$

In Quadrant 3, $x = \pi + 1.1960...$, or $4.3376...$

Verify by substituting each root in the given equation.

The solution is: $x \doteq 1.95 + 2\pi k, k \in \mathbb{Z}$ or $x \doteq 4.34 + 2\pi k, k \in \mathbb{Z}$

16. A student said that the identity $\sin 2\theta = 2 \sin \theta \cos \theta$ could be extended so that $\sin 4\theta = 4 \sin \theta \cos \theta$ and $\sin 6\theta = 6 \sin \theta \cos \theta$.

a) Is the student correct? If your answer is yes, explain why. If your answer is no, write correct identities for $\sin 4\theta$ and $\sin 6\theta$.

$\sin 4\theta = 4 \sin \theta \cos \theta$ is not an identity.

For example, suppose: $\theta = \frac{\pi}{3}$

$\sin 4\theta = \sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

$4 \sin \theta \cos \theta = 4 \sin \frac{\pi}{3} \cos \frac{\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \sqrt{3}$

$\sin 6\theta = 6 \sin \theta \cos \theta$ is not an identity.

For example, suppose: $\theta = \frac{\pi}{3}$

$\sin 6\theta = \sin \frac{6\pi}{3} = \sin 2\pi = 0$

$6 \sin \theta \cos \theta = 6 \sin \frac{\pi}{3} \cos \frac{\pi}{3} = 6\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{2}$

In $\sin 2\theta = 2 \sin \theta \cos \theta$, replace θ with 2θ:

$\sin 4\theta = 2 \sin 2\theta \cos 2\theta$

And, in $\sin 2\theta = 2 \sin \theta \cos \theta$, replace θ with 3θ:

$\sin 6\theta = 2 \sin 3\theta \cos 3\theta$

b) Write an identity for $\sin b\theta$, where b is a positive even number.

In $\sin 2\theta = 2 \sin \theta \cos \theta$, replace 2θ with $b\theta$, and replace θ with $\frac{b}{2}\theta$:

$\sin b\theta = 2 \sin \frac{b}{2}\theta \cos \frac{b}{2}\theta$

C

17. Prove each identity.

a) $-\sec 2x = \frac{\tan x + \cot x}{\tan x - \cot x}$

$$\text{R.S.} = \frac{\tan x + \cot x}{\tan x - \cot x} = \frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}} = \frac{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}}{\frac{\sin^2 x - \cos^2 x}{\sin x \cos x}} = \frac{\frac{1}{\sin x \cos x}}{\frac{-\cos 2x}{\sin x \cos x}} = \frac{1}{-\cos 2x} = -\sec 2x = \text{L.S.}$$

Since the left side is equal to the right side, the identity is proved.

b) $\cot 2x = \frac{\cot x - \tan x}{2}$

$$\text{R.S.} = \frac{\cot x - \tan x}{2} = \frac{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}}{2} = \frac{\frac{\cos^2 x - \sin^2 x}{\sin x \cos x}}{2} = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} = \frac{\cos 2x}{\sin 2x} = \cot 2x = \text{L.S.}$$

Since the left side is equal to the right side, the identity is proved.

18. For each equation, determine the solution over the set of real numbers.

a) $4 \sin 3x \cos 3x = 1$

$$\begin{aligned} 2(2 \sin 3x \cos 3x) &= 1 \\ 2 \sin 2(3x) &= 1 \\ 2 \sin 6x &= 1 \\ \sin 6x &= \frac{1}{2} \end{aligned}$$

$6x = \frac{\pi}{6} + 2\pi k, k \in \mathbb{Z}$, or

$6x = \frac{5\pi}{6} + 2\pi k, k \in \mathbb{Z}$

So, $x = \frac{\pi}{36} + \frac{\pi}{3}k, k \in \mathbb{Z}$, or

$x = \frac{5\pi}{36} + \frac{\pi}{3}k, k \in \mathbb{Z}$

b) $4 \cos^2 5x - 2 + \sqrt{3} = 0$

$$\begin{aligned} 2(2 \cos^2 5x - 1) &= -\sqrt{3} \\ 2 \cos 2(5x) &= -\sqrt{3} \\ \cos 10x &= -\frac{\sqrt{3}}{2} \end{aligned}$$

$10x = \frac{5\pi}{6} + 2\pi k, k \in \mathbb{Z}$, or

$10x = \frac{7\pi}{6} + 2\pi k, k \in \mathbb{Z}$

So, $x = \frac{\pi}{12} + \frac{\pi}{5}k, k \in \mathbb{Z}$, or

$x = \frac{7\pi}{60} + \frac{\pi}{5}k, k \in \mathbb{Z}$